A CONSTANT RANK THEOREM FOR PARTIAL CONVEX SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

CHUANQIANG CHEN

Abstract. Thanks to the test function of Bian-Guan[2], we successfully obtain a constant rank theorem for partial convex solutions of a class partial differential equations. This is the microscopic version of the macroscopic partial convexity principle in [1], and also is a generalization of the result in [2].

1. Introduction

The convex solution of partial differential equation is an interesting issue for a long time. And so far as we know, there are two important methods for this problem, which are macroscopic and microscopic methods. Whereas there are many solutions which are not convex. For example, the admissible solutions of the Hessian equations were studied in [7,10], the power concave solutions in [13,18], and the k-convex solutions in [11]. In this paper we will consider the partial convex solutions (see [1] or Definition 1.1 as below) of the elliptic and parabolic equations.

The study of macroscopic convexity is using a weak maximum principle, while the study of microscopic convexity is using a strong maximum principle. For the macroscopic convexity argument, Korevaar made breakthroughs in [14,15], he introduced concavity maximum principles for a class of quasilinear elliptic equations. And later it was improved by Kennington [13] and by Kawhol [12]. The theory further developed to its great generality by Alvarez-Lasry-Lions [1]. The key of the study of microscopic convexity is a method called *constant rank theorem* which was discovered in 2 dimension by Caffarelli-Friedman [5] (a similar result was also discovered by Singer-Wong-Yau-Yau [19] at the same time). Later the result in [5] was generalized to $\mathbb{R}^n$ by Korevaar-Lewis [17]. Recently the *constant rank theorem* was generalized to fully nonlinear equations in [6] and [2], where the result in [2] is the microscopic version of the macroscopic convexity principle in [1].

Constant rank theorem is a very useful tool to produce convex solutions in geometric analysis. By the corresponding homotopic deformation, the existence of convex solution comes from the *constant rank theorem*. For the geometric application of the *constant rank theorem*, the Christoffel-Minkowski problem and the related prescribing Weingarten curvature problems were studied in [8,9,10]. The preservation of convexity for the general geometric flows of hypersurfaces was given in [2]. Soon after the *constant rank theorem* for the level set was established in [3], where [3]

Research of the author was supported by Grant 10871187 from the National Natural Science Foundation of China.

is a microscopic version of [4] (also it was studied in [16]). And the existence of the k-convex hypersurface with prescribed mean curvature was given in [11] recently.

In this paper we consider the partial convexity of solutions of the following elliptic equation, and give a constant rank theorem for partial convex solutions

$$F(D^2u, Du, u, x) = 0, \quad x \in \Omega \subset \mathbb{R}^N, \tag{1.1}$$

where $F \in C^{2,1}(\mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega)$ and F is elliptic in the following sense

$$\left(\frac{\partial F}{\partial u_{ab}}(D^2u, Du, u, x)\right)_{N\times N} > 0, \quad \text{for all } x \in \Omega. \tag{1.2}$$

First, we give the definition of the partial convexity of a function u, which could be found in [1].

Definition 1.1. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$, where Ω is a domain in $\mathbb{R}^N = \mathbb{R}^{N'} \times \mathbb{R}^{N''}$, and N' and $N''$are two integers with $N = N' + N''$. Then u is partial convex (with respect to the first variable) is that $x' \to u(x', x'')$ is convex for every $x = (x', x'') \in \overline{\Omega}$. In particular, if $N'' = 0$, i.e. u is convex in $\Omega \subset \mathbb{R}^N = \mathbb{R}^{N'}$, u is said degenerate partial convex.

For simplicity, we introduce additional notations . As in [1], we denote $\mathcal{S}^n$ to be the set of all real symmetric $n \times n$ matrices. And we shall write $p \in \mathbb{R}^N$ in the form (p', p'') with $p' \in \mathbb{R}^{N'}$, $p'' \in \mathbb{R}^{N''}$ and split a matrix $A \in \mathcal{S}^N$ into $\begin{pmatrix} a & b \\ b^T & c \end{pmatrix}$ with $a \in \mathcal{S}^{N'}$, $b \in \mathbb{R}^{N' \times N''}$ and $c \in \mathcal{S}^{N''}$; we also let

$$F(A, p, u, x) = F(\begin{pmatrix} a & b \\ b^T & c \end{pmatrix}, p', p'', u, x', x''). \tag{1.3}$$

One of our main results is the following theorem

Theorem 1.2. *(CONSTANT RANK THEOREM) Suppose Ω is a domain in $\mathbb{R}^N = \mathbb{R}^{N'} \times \mathbb{R}^{N''}$ and $F(A, p, u, x) \in C^{2,1}(\mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega)$. If F satisfies (1.2) and the following condition*

$$F(\begin{pmatrix} a^{-1} & a^{-1}b \\ (a^{-1}b)^T & c + b^T a^{-1} b \end{pmatrix}, p', p'', u, x', x'') \text{ is locally convex in } (a, b, c, p'', u, x'). \tag{1.4}$$

If $u \in C^{2,1}(\Omega)$ is a partial convex solution of (1.1), then $(u_{ij})_{N'\times N'}$ has constant rank in Ω.

Remark 1.3. if $N'' = 0$, i.e. for the degenerate partial convexity, structure condition (1.4) is *inverse-convex* condition, the result of Bian-Guan [2]. And for general partial convexity, structure condition (1.4) is strictly stronger than the *inverse-convex* condition.

An immediate consequence of Theorem 1.2 is for partial convex solutions of the following quasi-linear second elliptic equation

$$\sum_{a,b=1}^{N} a^{ab}(x'', u_1(x), \cdots, u_{N'}(x)) u_{ab}(x) = f(x, u(x), Du(x)) > 0, \tag{1.5}$$

where $x \in \Omega \subset \mathbb{R}^N$ and

$$(a^{ab}(x'', u_1(x), \cdots, u_{N'}(x)))_{N\times N} > 0, \quad \text{for all } x \in \Omega. \tag{1.6}$$

Corollary 1.4. *Suppose Ω is a domain in $\mathbb{R}^N = \mathbb{R}^{N'} \times \mathbb{R}^{N''}$, and $u \in C^{2,1}(\Omega)$ is the partial convex solution of (1.5). If*

$$f(x', x'', u, p', p'') \text{ is locally concave in } (p'', u, x'),$$

then $(u_{ij})_{N'\times N'}$ has constant rank in Ω.

Set

$$F(D^2u, Du, u, x) = \sum_{a,b=1}^{N} a^{ab}(x'', u_1(x), \cdots, u_{N'}(x))u_{ab}(x) - f(x, u(x), Du(x)), \tag{1.7}$$

we can verify that F satisfies the structure condition (1.4) (see the equivalent condition (3.13) in the third section).

A corresponding result holds for the parabolic equation.

Theorem 1.5. *Suppose Ω is a domain in $\mathbb{R}^N = \mathbb{R}^{N'} \times \mathbb{R}^{N''}$, and $F(A, p, u, x, t) \in C^{2,1}(\mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \times (0, T])$. If F satisfies (1.2) for each t and the following condition*

$$F(\begin{pmatrix} a^{-1} & a^{-1}b \\ (a^{-1}b)^T & c + b^Ta^{-1}b \end{pmatrix}, p', p'', u, x', x'', t) \text{ is locally convex in } (a, b, c, p'', u, x'). \tag{1.8}$$

If $u \in C^{2,1}(\Omega \times (0, T])$ is a partial convex solution of the equation

$$\frac{\partial u}{\partial t} = F(D^2u, Du, u, x, t), \quad (x, t) \in \Omega \times (0, T], \tag{1.9}$$

then $(u_{ij}(x, t))_{N'\times N'}$ has constant rank in Ω for each $T \geqslant t > 0$. Moreover, let $l(t)$ be the minimal rank of $(u_{ij}(x, t))_{N'\times N'}$ in Ω, then $l(s) \leqslant l(t)$ for all $s \leqslant t \leqslant T$.

The rest of the paper is organized as follows. In section 2, we work on the Laplace equation, a special case of Corollary 1.4. In section 3, using the key auxiliary function $q(x)$ in [2], we do some preliminarily calculations on the constant rank theorem. In section 4, we prove the Theorem 1.2 using a strong maximum principle. In section 5, we prove Theorem 1.5. And the last section is devoted to a discussion of the structure condition.

Acknowledgement. The author would like to express sincere gratitude to Prof. Xi-Nan Ma for his encouragement and many suggestions in this subject.

2. AN EXAMPLE

In this section, we give a constant rank theorem for partial convex solutions of Laplace equation, a special case of Corollary 1.4.

We rewrite the result as follows.

Theorem 2.1. *Suppose Ω is a domain in $\mathbb{R}^N = \mathbb{R}^{N'} \times \mathbb{R}^{N''}$, and $u \in C^{2,1}(\Omega)$ is the partial convex solution of the following equation*

$$\Delta u(x) = \sum_{a=1}^{N} u_{aa}(x) = f(x, u(x), Du(x)) > 0, \quad x \in \Omega. \tag{2.1}$$

Assume

$$f(x', x'', u, p', p'') \text{ is locally concave in } (p'', u, x'), \tag{2.2}$$

then $(u_{ij})_{N'\times N'}$ *has constant rank in* Ω.

Before the proof of Theorem 2.1, we do some preliminaries. As in [9], we recall the definition of k-symmetric functions: For $1 \leqslant k \leqslant N'$, and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{N'}) \in \mathbb{R}^{N'}$,

$$\sigma_k(\lambda) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},$$

we denote by $\sigma_k(\lambda\,|i\,)$ the symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda\,|ij\,)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

The definition can be extended to symmetric matrices by letting $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), \lambda_2(W), \cdots, \lambda_{N'}(W))$ are the eigenvalues of the symmetric matrix W. We also set $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > N'$.

We need the following standard formulas, which could be found in [9], [2] or [3].

Lemma 2.2. *Suppose* $W = (W_{ij})$ *is diagonal, and* m *is positive integer, then*

$$\frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} \sigma_{m-1}(W\,|i\,), & if \quad i = j, \\ 0, & if \quad i \neq j. \end{cases}$$

$$\frac{\partial^2 \sigma_m(W)}{\partial W_{ij}\partial W_{kl}} = \begin{cases} \sigma_{m-2}(W\,|ik\,), & if \quad i = j, k = l, i \neq k, \\ -\sigma_{m-2}(W\,|ik\,), & if \quad i = l, j = k, i \neq j, \\ 0, & otherwise. \end{cases}$$

Proof of Theorem 2.1. With the assumptions of u in Theorem 2.1, u is automatically in $C^{3,1}$. We denote $W = (u_{ij})_{N'\times N'}$. For each $z_0 \in \Omega$ where W is of minimal rank l. We pick a small open neighborhood $\mathcal{O}$ of z_0, we will prove it always be rank of l in $\mathcal{O}$. We shall use the strong minimum principle to prove the theorem. Let

$$\phi(x) = \sigma_{l+1}(W), \tag{2.3}$$

then $\phi(z_0) = 0$. We shall show $\phi(x) \equiv 0$ in $\mathcal{O}$. If true, it implies the set $\{x \in \Omega | \phi(x) = 0\}$ is an open set. But it is also closed, then we get $\phi(x) \equiv 0$ in Ω since Ω connected, i.e. $(u_{ij})_{N'\times N'}$ is of constant rank l in Ω.

Following Caffarelli and Friedman [5], for two functions $h(y)$ and $k(y)$ defined in an open set $\mathcal{O} \subset \Omega$, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(h - k)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y). \tag{2.4}$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in the neighborhood $\mathcal{O}$, with the constants c_1 and c_2 independent of y in this neighborhood. Finally $h \sim k$ if $h \lesssim k$ and $k \lesssim h$.

We shall show that

$$\Delta\phi(x) = \sum_{a=1}^{N} \phi_{aa}(x) \lesssim 0. \tag{2.5}$$

Since $\phi(x) \geqslant 0$ in Ω and $\phi(z_0) = 0$, it then follows from the Strong Minimum Principle that $\phi(x) \equiv 0$ in $\mathcal{O}$.

For any fixed point $x \in \mathcal{O}$, we rotate coordinate $e_1, \cdots, e_{N'}$ such that the matrix $u_{ij}, i, j = 1, \cdots, N'$ is diagonal and without loss of generality we assume $u_{11} \leqslant u_{22} \leqslant \cdots \leqslant u_{N'N'}$. Then there is a positive constant $C > 0$ depending only on $\|u\|_{C^{3,1}}$ and $\mathcal{O}$, such that $u_{N'N'} \geqslant \cdots \geqslant u_{N'-l+1N'-l+1} \geqslant C > 0$ for all $x \in \mathcal{O}$. For convenience we denote $G = \{N'-l+1, \cdots, N'\}$ and $B = \{1, 2, \cdots, N'-l\}$ which means good terms and bad ones in indices respectively. Without confusion we will also simply denote $B = \{u_{11}, \cdots, u_{N'-lN'-l}\}$ and $G = \{u_{N'-l+1N'-l+1}, \cdots, u_{N'N'}\}$. In the following, all the calculation at the point x are using the relation $\lesssim$ with the understanding that the constants in (2.4) are under control.

Following a direct computation as in [9] and W is diagonal, we can get

$$0 \sim \phi \sim \sigma_l(G) \sum_{i \in B} u_{ii}, \text{ and } u_{ii} \sim 0 \text{ for each } i \in B; \tag{2.6}$$

$$0 \sim \phi_a \sim \sigma_l(G) \sum_{i \in B} u_{iia}; \tag{2.7}$$

then by (2.6),(2.7) and Lemma 2.2, we obtain

$$\begin{aligned}
\Delta\phi = \sum_{a=1}^{N} \frac{\partial^2 \phi}{\partial x_a \partial x_a} &= \sum_{a=1}^{N} [\sum_{i,j=1}^{N'} \frac{\partial \sigma_{l+1}(W)}{\partial u_{ij}} u_{ijaa} + \sum_{i,j,k,l=1}^{N'} \frac{\partial^2 \sigma_{l+1}(W)}{\partial u_{ij} \partial u_{kl}} u_{ija} u_{kla}] \\
&= \sum_{a=1}^{N} [\sum_{i=1}^{N'} \frac{\partial \sigma_{l+1}(W)}{\partial u_{ii}} u_{iiaa} + \sum_{i,j=1}^{N'} \frac{\partial^2 \sigma_{l+1}(W)}{\partial u_{ii} \partial u_{jj}} u_{iia} u_{jja} + \sum_{i,j=1}^{N'} \frac{\partial^2 \sigma_{l+1}(W)}{\partial u_{ij} \partial u_{ji}} u_{ija} u_{jia}] \\
&\sim \sum_{a=1}^{N} [\sigma_l(G) \sum_{i \in B} u_{iiaa} - 2\sigma_l(G) \sum_{i \in B, j \in G} \frac{1}{u_{jj}} u_{ija} u_{jia}] \\
&\sim \sigma_l(G) \sum_{i \in B} (\Delta u)_{ii} - 2\sigma_l(G) \sum_{a=1}^{N} \sum_{i \in B, j \in G} \frac{1}{u_{jj}} u_{ija}^2.
\end{aligned} \tag{2.8}$$

For each $i \in B$, we differentiate (2.1) twice in x_i, then we obtain

$$\begin{aligned}
(\Delta u)_{ii} =& [f_{x_i} + f_u u_i + \sum_{a=1}^{N} f_{p_a} u_{ai}]_i \\
=& f_{x_i x_i} + 2 f_{u,x_i} u_i + f_{u,u} u_i^2 \\
&+ 2 \sum_{a=1}^{N} f_{x_i, p_a} u_{ai} + 2 \sum_{a=1}^{N} f_{u,p_a} u_i u_{ai} + \sum_{a,b=1}^{N} f_{p_a, p_b} u_{ai} u_{bi},
\end{aligned}$$

since $W = (u_{ij})_{N' \times N'}$ is diagonal and (2.6), we get from the above equation

$$\begin{aligned}
(\Delta u)_{ii} \sim& f_{x_i x_i} + 2 f_{u,x_i} u_i + f_{u,u} u_i^2 \\
&+ 2 \sum_{\alpha=N'+1}^{N} f_{x_i, p_\alpha} u_{\alpha i} + 2 \sum_{\alpha=N'+1}^{N} f_{u,p_\alpha} u_i u_{\alpha i} + \sum_{\alpha,\beta=N'+1}^{N} f_{p_\alpha, p_\beta} u_{\alpha i} u_{\beta i},
\end{aligned} \tag{2.9}$$

so we obtain from (2.8) and (2.9)

$$
\begin{aligned}
\frac{\Delta\phi}{\sigma_l(G)} \sim & \sum_{i=1}^{N'-l}(\Delta u)_{ii} - 2\sum_{a=1}^{N}\sum_{j=N'-l+1}^{N'}\frac{1}{u_{jj}}\sum_{i=1}^{N'-l}u_{ija}^2 \\
\sim & -2\sum_{a=1}^{N}\sum_{j=N'-l+1}^{N'}\frac{1}{u_{jj}}\sum_{i=1}^{N'-l}u_{ija}^2 + \sum_{i=1}^{N'-l}[f_{x_ix_i} + 2f_{u,x_i}u_i + f_{u,u}u_i^2 \\
& + 2\sum_{\alpha=N'+1}^{N} f_{x_i,p_\alpha}u_{\alpha i} + 2\sum_{\alpha=N'+1}^{N} f_{u,p_\alpha}u_iu_{\alpha i} + \sum_{\alpha,\beta=N'+1}^{N} f_{p_\alpha,p_\beta}u_{\alpha i}u_{\beta i}].
\end{aligned} \tag{2.10}
$$

By the condition (2.2), we obtain (2.5). The proof of Theorem 2.1 is completed.

Remark 2.3. In (2.8), we have used Lemma 2.5 in [2], otherwise the first "$\sim$" will be "$\lesssim$".

Remark 2.4. By a similar proof as above, we can get the general case of Corollary 1.4.

3. PRIMARILY CALCULATIONS ON THE CONSTANT RANK THEOREM

3.1. **calculations on the test function.** With the assumptions in Theorem 1.2 and Theorem 1.5, u is in $C^{3,1}$. Let $W = (u_{ij})_{N'\times N'}$ and $l = \min_{x\in\Omega} rank(W(x))$. We may assume $l \leqslant N'-1$, otherwise there is nothing to prove. Suppose $z_0 \in \Omega$ is a point where W is of minimal rank l.

Throughout this paper we assume that $1 \leqslant i,j,k,l,m,n \leqslant N'$, $N' \leqslant \alpha,\beta,\gamma,\eta,\xi,\zeta \leqslant N''$, $1 \leqslant a,b,c,d \leqslant N$ and $\sigma_j(W) = 0$ if $j < 0$ or $j > N'$. As in Bian-Guan [2],we define for $W = (u_{ij}(x)) \in \mathcal{S}^{N'}$,

$$
q(W) = \begin{cases} \frac{\sigma_{l+2}(W)}{\sigma_{l+1}(W)}, & if \quad \sigma_{l+1}(W) > 0, \\ 0, & if \quad \sigma_{l+1}(W) = 0. \end{cases} \tag{3.1}
$$

and we consider the following test function

$$
\phi = \sigma_{l+1}(W) + q(W). \tag{3.2}
$$

For each $z_0 \in \Omega$ where W is of minimal rank l. We pick an open neighborhood $\mathcal{O}$ of z_0, and for any fixed point $x \in \mathcal{O}$, we rotate coordinate $e_1, \cdots, e_{N'}$ such that the matrix $u_{ij}, i,j = 1,\cdots,N'$ is diagonal and without loss of generality we assume $u_{11} \leqslant u_{22} \leqslant \cdots \leqslant u_{N'N'}$. Then there is a positive constant $C > 0$ depending only on $\|u\|_{C^{3,1}}$ and $\mathcal{O}$, such that $u_{N'N'} \geqslant \cdots \geqslant u_{N'-l+1N'-l+1} \geqslant C > 0$ for all $x \in \mathcal{O}$. For convenience we denote $G = \{N'-l+1,\cdots,N'\}$ and $B = \{1,2,\cdots,N'-l\}$ which means good terms and bad ones in indices respectively. Without confusion we will also simply denote $B = \{u_{11},\cdots,u_{N'-lN'-l}\}$ and $G = \{u_{N'-l+1N'-l+1},\cdots,u_{N'N'}\}$. Note that for any $\delta > 0$, we may choose $\mathcal{O}$ small enough such that $u_{jj} < \delta$ for all $j \in B$ and $x \in \mathcal{O}$.

We will use notation $h = O(f)$ if $|h(x)| \leqslant Cf(x)$ for $x \in \mathcal{O}$ with positive constant C under control. It is clear that $u_{ii} = O(\phi)$ for all $i \in B$.

To get around $\sigma_{l+1}(W) = 0$, for $\varepsilon > 0$ sufficient small, we consider

$$
q(W_\varepsilon) = \frac{\sigma_{l+2}(W_\varepsilon)}{\sigma_{l+1}(W_\varepsilon)}, \quad \phi_\varepsilon = \sigma_{l+1}(W_\varepsilon) + q(W_\varepsilon), \tag{3.3}
$$

where $W_\varepsilon = W+\varepsilon I$. We will also denote $B_\varepsilon = \{u_{11}+\varepsilon, \cdots, u_{N'-lN'-l}+\varepsilon\}$, $G_\varepsilon = \{u_{N'-l+1N'-l+1}+\varepsilon, \cdots, u_{N'N'}+\varepsilon\}$. (see Bian-Guan[2]).

Set $u_\varepsilon(x) = u(x) + \frac{\varepsilon}{2}|x'|^2$, then $W_\varepsilon = ((u_\varepsilon)_{ij})_{N'\times N'}$. To simplify the notations, we will write u for u_ε, q for q_ε, W for W_ε, G for G_ε, and B for B_ε with the understanding that all the estimates will be independent of ε. In this setting, if we pick $\mathcal{O}$ small enough, there is $C > 0$ independent of ε such that

$$\phi \geqslant C\varepsilon, \sigma_1(B) \geqslant C\varepsilon, \text{ for all } x \in \mathcal{O}. \tag{3.4}$$

First, we consider the regularity of $q(W(x))$.

Proposition 3.1. *([2]) let $u \in C^{3,1}(\Omega)$ be a partial convex function with the first variable and $W(x) = (u_{ij}(x))_{N'\times N'}$. Let $l = \min_{x\in\Omega} rank(W(x))$, then the function $q(x) = q(W(x))$ defined in (3.1) is in $C^{1,1}(\Omega)$.*

In the following, we denote

$$\begin{aligned}
&F^{ab} = \frac{\partial F}{\partial u_{ab}}, F^{p_a} = \frac{\partial F}{\partial u_a}, F^u = \frac{\partial F}{\partial u},\\
&F^{ab,cd} = \frac{\partial^2 F}{\partial u_{ab}\partial u_{cd}}, F^{ab,p_c} = \frac{\partial^2 F}{\partial u_{ab}\partial u_c}, F^{ab,u} = \frac{\partial^2 F}{\partial u_{ab}\partial u},\\
&F^{p_a p_b} = \frac{\partial^2 F}{\partial u_a\partial u_b}, F^{p_a,u} = \frac{\partial^2 F}{\partial u_a\partial u}, F^{u,u} = \frac{\partial^2 F}{\partial u\partial u},
\end{aligned}$$

where $1 \leqslant a, b, c \leqslant N$.

Theorem 3.2. *Suppose Ω is a domain in $\mathbb{R}^N = \mathbb{R}^{N'} \times \mathbb{R}^{N''}$ and $u \in C^{3,1}(\Omega)$ is a partial convex solution of (1.1). Let l be the minimal rank of $W = (u_{ij})_{N'\times N'}$ in Ω. Suppose l is attained at $z_0 \in \Omega$, and $\mathcal{O}$ is a small neighborhood of z_0 as above. For any fixed point $x \in \mathcal{O}$ we choose the coordinate such that $W(x)$ is diagonal. Then at x we have*

$$\begin{aligned}
\sum_{a,b=1}^N F^{ab}\phi_{ab} &= \sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B\,|i) - \sigma_2(B\,|i)}{\sigma_1^2(B)}]\sum_{a,b=1}^N F^{ab}u_{iiab}\\
&-2\sum_{i\in B, j\in G}[\sigma_l(G) + \frac{\sigma_1^2(B\,|i) - \sigma_2(B\,|i)}{\sigma_1^2(B)}]\frac{1}{u_{jj}}\sum_{a,b=1}^N F^{ab}u_{ija}u_{ijb}\\
&-\frac{1}{\sigma_1^3(B)}\sum_{i\in B}\sum_{a,b=1}^N F^{ab}[\sigma_1(B)u_{iia} - u_{ii}\sum_{j\in B}u_{jja}][\sigma_1(B)u_{iib} - u_{ii}\sum_{j\in B}u_{jjb}]\\
&-\frac{1}{\sigma_1(B)}\sum_{\substack{i,j\in B\\ i\neq j}}\sum_{a,b=1}^N F^{ab}u_{ija}u_{ijb}\\
&+O(\sum_{i,j\in B}|\nabla u_{ij}| + \phi).
\end{aligned} \tag{3.5}$$

In fact, u in (3.5) is $u_\varepsilon(x) = u(x) + \frac{\varepsilon}{2}|x'|^2$ defined as above (we omit the subindex ε).

Proof. The proof is similar to the proof in [2]. We give the main process.

Following the assumptions as above, and for a similar computation as in [2], we have

$$\sigma_1(B) = O(\phi), u_{ii} = O(\phi) \text{ for every } i \in B. \tag{3.6}$$

Since $\phi(x) = \sigma_{l+1}(W) + q(W)$, then by the chain rule we have

$$\begin{aligned}
\sum_{a,b=1}^{N} F^{ab}\phi_{ab} =& \sum_{a,b=1}^{N} F^{ab}[\sum_{i,j} \frac{\partial \phi}{\partial u_{ij}} u_{ijab} + \sum_{i,j,k,l} \frac{\partial^2 \phi}{\partial u_{ij}\partial u_{kl}} u_{ija}u_{klb}] \\
=& \sum_{a,b=1}^{N} F^{ab} \sum_{i,j} [\frac{\partial \sigma_{l+1}(W)}{\partial u_{ij}} + \frac{\partial q(W)}{\partial u_{ij}}] u_{ijab} \\
&+ \sum_{a,b=1}^{N} F^{ab} \sum_{i,j,k,l} [\frac{\partial^2 \sigma_{l+1}(W)}{\partial u_{ij}\partial u_{kl}} + \frac{\partial^2 q(W)}{\partial u_{ij}\partial u_{kl}}] u_{ija}u_{rsb}.
\end{aligned} \tag{3.7}$$

Since W is diagonal and by lemma 2.2, the first term on the right hand side of (3.7) is

$$\begin{aligned}
\sum_{a,b=1}^{N} F^{ab} \sum_{i,j} \frac{\partial \sigma_{l+1}(W)}{\partial u_{ij}} u_{ijab} =& \sum_{a,b=1}^{N} F^{ab} \sum_{i} \frac{\partial \sigma_{l+1}(W)}{\partial u_{ii}} u_{iiab} \\
=& \sum_{a,b=1}^{N} F^{ab} \sum_{i \in B} \sigma_l(G) u_{iiab} + O(\phi).
\end{aligned} \tag{3.8}$$

Using Lemma 2.4 in [2], the second term on the right hand side of (3.7) is

$$\begin{aligned}
\sum_{a,b=1}^{N} F^{ab} \sum_{i,j} \frac{\partial q(W)}{\partial u_{ij}} u_{ijab} =& \sum_{a,b=1}^{N} F^{ab} \sum_{i} \frac{\partial q(W)}{\partial u_{ii}} u_{iiab} \\
=& \sum_{a,b=1}^{N} F^{ab} \sum_{i \in B} \frac{\sigma_1^2(B\,|i) - \sigma_2(B\,|i)}{\sigma_1^2(B)} u_{iiab} + O(\phi).
\end{aligned} \tag{3.9}$$

As in [2], the third term on the right hand side of (3.7) is

$$\begin{aligned}
&\sum_{a,b=1}^{N} F^{ab} \sum_{i,j,k,l} \frac{\partial^2 \sigma_{l+1}(W)}{\partial u_{ij}\partial u_{kl}} u_{ija}u_{klb} \\
=& \sum_{a,b=1}^{N} F^{ab} [\sum_{i \neq j} \frac{\partial^2 \sigma_{l+1}(W)}{\partial u_{ii}\partial u_{jj}} u_{iia}u_{jjb} + \sum_{i \neq j} \frac{\partial^2 \sigma_{l+1}(W)}{\partial u_{ij}\partial u_{ji}} u_{ija}u_{jib}] \\
=& -2 \sum_{a,b=1}^{N} F^{ab} \sum_{i \in B, j \in G} \sigma_{l-1}(G\,|j) u_{ija}u_{jib} + O(\sum_{i,j \in B} |\nabla u_{ij}| + \phi).
\end{aligned} \tag{3.10}$$

From Proposition 2.1 in [2], we can get

$$
\begin{aligned}
&\sum_{i,j,k,l}\frac{\partial^2 q(W)}{\partial u_{ij}\partial u_{kl}}u_{ija}u_{klb}=-2\sum_{i\in B,j\in G}\frac{\sigma_1^2(B\,|i)-\sigma_2(B\,|i)}{\sigma_1^2(B)u_{jj}}u_{ija}u_{ijb}\\
&-\frac{1}{\sigma_1^3(B)}\sum_{i\in B}[\sigma_1(B)u_{iia}-u_{ii}\sum_{j\in B}u_{jja}][\sigma_1(B)u_{iib}-u_{ii}\sum_{j\in B}u_{jjb}]\\
&-\frac{1}{\sigma_1(B)}\sum_{\substack{i,j\in B\\ i\neq j}}u_{ija}u_{ijb}+O(\sum_{i,j\in B}|\nabla u_{ij}|+\phi).
\end{aligned}
\tag{3.11}
$$

So the fourth term on the right hand side of (3.7) is

$$
\begin{aligned}
&\sum_{a,b=1}^{N}F^{ab}\sum_{i,j,k,l}\frac{\partial^2 q(W)}{\partial u_{ij}\partial u_{kl}}u_{ija}u_{klb}\\
=&-2\sum_{a,b=1}^{N}F^{ab}\sum_{i\in B,j\in G}\frac{\sigma_1^2(B\,|i)-\sigma_2(B\,|i)}{\sigma_1^2(B)u_{jj}}u_{ija}u_{ijb}\\
&-\frac{1}{\sigma_1^3(B)}\sum_{a,b=1}^{N}F^{ab}\sum_{i\in B}[\sigma_1(B)u_{iia}-u_{ii}\sum_{j\in B}u_{jja}][\sigma_1(B)u_{iib}-u_{ii}\sum_{j\in B}u_{jjb}]\\
&-\frac{1}{\sigma_1(B)}\sum_{a,b=1}^{N}F^{ab}\sum_{\substack{i,j\in B\\ i\neq j}}u_{ija}u_{ijb}+O(\sum_{i,j\in B}|\nabla u_{ij}|+\phi).
\end{aligned}
\tag{3.12}
$$

Substitute (3.8), (3.9), (3.10) and (3.12) into (3.7), then we obtain(3.5).

3.2. **calculation on structure condition.** Now we discuss the structure condition (1.4). We write $A=\begin{pmatrix} a & b\\ b^T & c\end{pmatrix}$ and $a^{-1}=(a^{ij})$, where $a=(a_{ij})\in\mathcal{S}^{N'}$, $b=(b_{k\alpha})\in\mathbb{R}^{N'\times N''}$ and $c=(c_{\alpha\beta})\in\mathcal{S}^{N''}$.

Lemma 3.3. *The condition (1.4) is equivalent to*

$$
\begin{aligned}
&\sum_{a,b,c,d=1}^{N}F^{ab,cd}(A,p,u,x)X_{ab}X_{cd}+2\sum_{a,b=1}^{N}\sum_{k,l=1}^{N'}F^{ab}a^{kl}X_{ka}X_{lb}+2\sum_{a,b=1}^{N}\sum_{\alpha=N'+1}^{N}F^{ab,p_\alpha}X_{ab}X_\alpha\\
&+2\sum_{a,b=1}^{N}F^{ab,u}X_{ab}Y+2\sum_{a,b=1}^{N}\sum_{i=1}^{N'}F^{ab,x_i}X_{ab}Z_i+\sum_{\alpha,\beta=N'+1}^{N}F^{p_\alpha,p_\beta}X_\alpha X_\beta+2\sum_{\alpha=N'+1}^{N}F^{p_\alpha,u}X_\alpha Y\\
&+2\sum_{\alpha=N'+1}^{N}\sum_{i=1}^{N'}F^{p_\alpha,x_i}X_\alpha Z_i+F^{u,u}Y^2+2\sum_{i=1}^{N'}F^{u,x_i}YZ_i+\sum_{i,j=1}^{N'}F^{x_i,x_j}Z_iZ_j\geqslant 0,
\end{aligned}
\tag{3.13}
$$

for every $\widetilde{X}=((X_{ab}),(X_\alpha),Y,(Z_i))\in\mathcal{S}^N\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}$.

Proof. We denote $G(a,b,c,p'',u,x')=F(\begin{pmatrix} a^{-1} & a^{-1}b\\ (a^{-1}b)^T & c+b^Ta^{-1}b\end{pmatrix},p',p'',u,x',x'')$, and we let $1\leqslant i,j,k,l,m,n,s,t\leqslant N'$, $N'\leqslant\alpha,\beta,\gamma,\eta,\xi,\zeta\leqslant N''$, and $1\leqslant a,b,c,d\leqslant N$. Then condition (1.4)

is equivalent to

(3.14)

$$\begin{aligned}&\sum_{i,j,k,l}\frac{\partial^2 G}{\partial a_{ij}\partial a_{kl}}X_{ij}X_{kl}+2\sum_{i,j,k,\alpha}\frac{\partial^2 G}{\partial a_{ij}\partial b_{k\alpha}}X_{ij}Y_{k\alpha}+2\sum_{i,j,\alpha,\beta}\frac{\partial^2 G}{\partial a_{ij}\partial c_{\alpha\beta}}X_{ij}Z_{\alpha\beta}+2\sum_{i,j,\alpha}\frac{\partial^2 G}{\partial a_{ij}\partial p_{\alpha}}X_{ij}X_{\alpha}\\&+2\sum_{i,j}\frac{\partial^2 G}{\partial a_{ij}\partial u}X_{ij}Y+2\sum_{i,j,k}\frac{\partial^2 G}{\partial a_{ij}\partial x_k}X_{ij}Z_k+\sum_{k,\alpha,l,\beta}\frac{\partial^2 G}{\partial b_{k\alpha}\partial b_{l\beta}}Y_{k\alpha}Y_{l\beta}+2\sum_{k,\alpha,\gamma,\eta}\frac{\partial^2 G}{\partial b_{k\alpha}\partial c_{\gamma\eta}}Y_{k\alpha}Z_{\gamma\eta}\\&+2\sum_{k,\alpha,\beta}\frac{\partial^2 G}{\partial b_{k\alpha}\partial p_{\beta}}Y_{k\alpha}X_{\beta}+2\sum_{k,\alpha}\frac{\partial^2 G}{\partial b_{k\alpha}\partial u}Y_{k\alpha}Y+2\sum_{k,\alpha,l}\frac{\partial^2 G}{\partial b_{k\alpha}\partial x_l}Y_{k\alpha}Z_l+\sum_{\alpha,\beta,\gamma,\eta}\frac{\partial^2 G}{\partial c_{\alpha\beta}\partial c_{\gamma\eta}}Z_{\alpha\beta}Z_{\gamma\eta}\\&+2\sum_{\alpha,\beta,\gamma}\frac{\partial^2 G}{\partial c_{\alpha\beta}\partial p_{\gamma}}Z_{\alpha\beta}X_{\gamma}+2\sum_{\alpha,\beta}\frac{\partial^2 G}{\partial c_{\alpha\beta}\partial u}Z_{\alpha\beta}Y+2\sum_{\alpha,\beta,l}\frac{\partial^2 G}{\partial c_{\alpha\beta}\partial x_l}Z_{\alpha\beta}Z_l+\sum_{\alpha,\gamma}\frac{\partial^2 G}{\partial p_{\alpha}\partial p_{\gamma}}X_{\alpha}X_{\gamma}\\&+2\sum_{\alpha}\frac{\partial^2 G}{\partial p_{\alpha}\partial u}X_{\alpha}Y+2\sum_{\alpha,l}\frac{\partial^2 G}{\partial p_{\alpha}\partial x_l}X_{\alpha}Z_l+\frac{\partial^2 G}{\partial u\partial u}Y^2+2\sum_{l}\frac{\partial^2 G}{\partial u\partial x_l}YZ_l+\sum_{k,l}\frac{\partial^2 G}{\partial x_k\partial x_l}Z_kZ_l\geqslant 0,\end{aligned}$$

for every $((X_{ij}),(Y_{k\alpha}),(Z_{\alpha\beta}),(X_\alpha),Y,(Z_i))\in\mathcal{S}^{N'}\times\mathbb{R}^{N'\times N''}\times\mathcal{S}^{N''}\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}$.

To get the equivalent condition (3.13), we shall represent all the derivatives of G in (3.14) by the derivatives of F.

Suppose $a^{-1}b=(B_{k\alpha})=(\sum_l a^{kl}b_{l\alpha})$, and $c+b^Ta^{-1}b=(C_{\alpha\beta})=(c_{\alpha\beta}+\sum_{k,l}b_{k\alpha}a^{kl}b_{l\beta})$, then

$$G(a,b,c,p'',u,x')=F(\begin{pmatrix}a^{-1} & (B_{k\alpha})\\ (B_{k\alpha})^T & (C_{\alpha\beta})\end{pmatrix},p',p'',u,x',x'').$$

A direct computation yields

$$\frac{\partial G}{\partial a_{ij}}=\sum_{k,l}F^{kl}\frac{\partial a^{kl}}{\partial a_{ij}}+\sum_{k,\beta}F^{k\beta}\frac{\partial B_{k\beta}}{\partial a_{ij}}+\sum_{\alpha,l}F^{\alpha l}\frac{\partial B_{l\alpha}}{\partial a_{ij}}+\sum_{\alpha,\beta}F^{\alpha\beta}\frac{\partial C_{\alpha\beta}}{\partial a_{ij}},\tag{3.15}$$

$$\frac{\partial G}{\partial b_{k\beta}}=\sum_{m,\alpha}F^{m\alpha}\frac{\partial B_{m\alpha}}{\partial b_{k\beta}}+\sum_{\alpha,n}F^{\alpha n}\frac{\partial B_{n\alpha}}{\partial b_{k\beta}}+\sum_{\gamma,\eta}F^{\gamma\eta}\frac{\partial C_{\gamma\eta}}{\partial b_{k\beta}}.\tag{3.16}$$

So we have the second derivatives of G in (3.14) as follows. The derivatives of G in the last ten terms are simple,

$$\begin{aligned}&\frac{\partial^2 G}{\partial x_k\partial x_l}=F^{x_k,x_l},\frac{\partial^2 G}{\partial u\partial x_l}=F^{u,x_l},\frac{\partial^2 G}{\partial u\partial u}=F^{u,u},\\&\frac{\partial^2 G}{\partial p_\alpha\partial x_l}=F^{p_\alpha,x_l},\frac{\partial^2 G}{\partial p_\alpha\partial u}=F^{p_\alpha,u},\frac{\partial^2 G}{\partial p_\alpha\partial p_\gamma}=F^{p_\alpha,p_\gamma},\\&\frac{\partial^2 G}{\partial c_{\alpha\beta}\partial x_l}=F^{\alpha\beta,x_l},\frac{\partial^2 G}{\partial c_{\alpha\beta}\partial u}=F^{\alpha\beta,u},\\&\frac{\partial^2 G}{\partial c_{\alpha\beta}\partial p_\gamma}=F^{\alpha\beta,p_\gamma},\frac{\partial^2 G}{\partial c_{\alpha\beta}\partial c_{\gamma\eta}}=F^{\alpha\beta,\gamma\eta}.\end{aligned}$$

From (3.15), we can get the derivatives of G in the third-sixth terms of (3.14)

$$\frac{\partial^2 G}{\partial a_{ij}\partial c_{\gamma\eta}}=\sum_{k,l}F^{kl,\gamma\eta}\frac{\partial a^{kl}}{\partial a_{ij}}+\sum_{k,\beta}F^{k\beta,\gamma\eta}\frac{\partial B_{k\beta}}{\partial a_{ij}}+\sum_{\alpha,l}F^{\alpha l,\gamma\eta}\frac{\partial B_{l\alpha}}{\partial a_{ij}}+\sum_{\alpha,\beta}F^{\alpha\beta,\gamma\eta}\frac{\partial C_{\alpha\beta}}{\partial a_{ij}},$$

$$\frac{\partial^2 G}{\partial a_{ij}\partial p_\gamma}=\sum_{k,l}F^{kl,p_\gamma}\frac{\partial a^{kl}}{\partial a_{ij}}+\sum_{k,\beta}F^{k\beta,p_\gamma}\frac{\partial B_{k\beta}}{\partial a_{ij}}+\sum_{\alpha,l}F^{\alpha l,p_\gamma}\frac{\partial B_{l\alpha}}{\partial a_{ij}}+\sum_{\alpha,\beta}F^{\alpha\beta,p_\gamma}\frac{\partial C_{\alpha\beta}}{\partial a_{ij}},$$

$$\frac{\partial^2 G}{\partial a_{ij}\partial u}=\sum_{k,l}F^{kl,u}\frac{\partial a^{kl}}{\partial a_{ij}}+\sum_{k,\beta}F^{k\beta,u}\frac{\partial B_{k\beta}}{\partial a_{ij}}+\sum_{\alpha,l}F^{\alpha l,u}\frac{\partial B_{l\alpha}}{\partial a_{ij}}+\sum_{\alpha,\beta}F^{\alpha\beta,u}\frac{\partial C_{\alpha\beta}}{\partial a_{ij}},$$

$$\frac{\partial^2 G}{\partial a_{ij}\partial x_m}=\sum_{k,l}F^{kl,x_m}\frac{\partial a^{kl}}{\partial a_{ij}}+\sum_{k,\beta}F^{k\beta,x_m}\frac{\partial B_{k\beta}}{\partial a_{ij}}+\sum_{\alpha,l}F^{\alpha l,x_m}\frac{\partial B_{l\alpha}}{\partial a_{ij}}+\sum_{\alpha,\beta}F^{\alpha\beta,x_m}\frac{\partial C_{\alpha\beta}}{\partial a_{ij}}.$$

From (3.16), we can get the derivatives of G in the eighth-eleventh terms of (3.14)

$$\frac{\partial^2 G}{\partial b_{k\beta}\partial c_{\xi\zeta}}=\sum_{m,\alpha}F^{m\alpha,\xi\zeta}\frac{\partial B_{m\alpha}}{\partial b_{k\beta}}+\sum_{\alpha,n}F^{\alpha n,\xi\zeta}\frac{\partial B_{n\alpha}}{\partial b_{k\beta}}+\sum_{\gamma,\eta}F^{\gamma\eta,\xi\zeta}\frac{\partial C_{\gamma\eta}}{\partial b_{k\beta}},$$

$$\frac{\partial^2 G}{\partial b_{k\beta}\partial p_\zeta}=\sum_{m,\alpha}F^{m\alpha,p_\zeta}\frac{\partial B_{m\alpha}}{\partial b_{k\beta}}+\sum_{\alpha,n}F^{\alpha n,p_\zeta}\frac{\partial B_{n\alpha}}{\partial b_{k\beta}}+\sum_{\gamma,\eta}F^{\gamma\eta,p_\zeta}\frac{\partial C_{\gamma\eta}}{\partial b_{k\beta}},$$

$$\frac{\partial^2 G}{\partial b_{k\beta}\partial u}=\sum_{m,\alpha}F^{m\alpha,u}\frac{\partial B_{m\alpha}}{\partial b_{k\beta}}+\sum_{\alpha,n}F^{\alpha n,u}\frac{\partial B_{n\alpha}}{\partial b_{k\beta}}+\sum_{\gamma,\eta}F^{\gamma\eta,u}\frac{\partial C_{\gamma\eta}}{\partial b_{k\beta}},$$

$$\frac{\partial^2 G}{\partial b_{k\beta}\partial x_i}=\sum_{m,\alpha}F^{m\alpha,x_i}\frac{\partial B_{m\alpha}}{\partial b_{k\beta}}+\sum_{\alpha,n}F^{\alpha n,x_i}\frac{\partial B_{n\alpha}}{\partial b_{k\beta}}+\sum_{\gamma,\eta}F^{\gamma\eta,x_i}\frac{\partial C_{\gamma\eta}}{\partial b_{k\beta}}.$$

Also from (3.15) we can get the derivative of G in the first term of (3.14)

$$\begin{aligned}\frac{\partial^2 G}{\partial a_{ij}\partial a_{mn}}=&\sum_{k,l}F^{kl}\frac{\partial^2 a^{kl}}{\partial a_{ij}\partial a_{mn}}+\sum_{k,\beta}F^{k\beta}\frac{\partial^2 B_{k\beta}}{\partial a_{ij}\partial a_{mn}}+\sum_{\alpha,l}F^{\alpha l}\frac{\partial^2 B_{l\alpha}}{\partial a_{ij}\partial a_{mn}}+\sum_{\alpha,\beta}F^{\alpha\beta}\frac{\partial^2 C_{\alpha\beta}}{\partial a_{ij}\partial a_{mn}}\\
&+\sum_{k,l}\frac{\partial a^{kl}}{\partial a_{ij}}[\sum_{s,t}F^{kl,st}\frac{\partial a^{st}}{\partial a_{mn}}+\sum_{s,\eta}F^{kl,s\eta}\frac{\partial B_{s\eta}}{\partial a_{mn}}+\sum_{\gamma,t}F^{kl,\gamma t}\frac{\partial B_{t\gamma}}{\partial a_{mn}}+\sum_{\gamma,\eta}F^{kl,\gamma\eta}\frac{\partial C_{\gamma\eta}}{\partial a_{mn}}]\\
&+\sum_{k,\beta}\frac{\partial B_{k\beta}}{\partial a_{ij}}[\sum_{s,t}F^{k\beta,st}\frac{\partial a^{st}}{\partial a_{mn}}+\sum_{s,\eta}F^{k\beta,s\eta}\frac{\partial B_{s\eta}}{\partial a_{mn}}+\sum_{\gamma,t}F^{k\beta,\gamma t}\frac{\partial B_{t\gamma}}{\partial a_{mn}}+\sum_{\gamma,\eta}F^{k\beta,\gamma\eta}\frac{\partial C_{\gamma\eta}}{\partial a_{mn}}]\\
&+\sum_{\alpha,l}\frac{\partial B_{l\alpha}}{\partial a_{ij}}[\sum_{s,t}F^{\alpha l,st}\frac{\partial a^{st}}{\partial a_{mn}}+\sum_{s,\eta}F^{\alpha l,s\eta}\frac{\partial B_{s\eta}}{\partial a_{mn}}+\sum_{\gamma,t}F^{\alpha l,\gamma t}\frac{\partial B_{t\gamma}}{\partial a_{mn}}+\sum_{\gamma,\eta}F^{\alpha l,\gamma\eta}\frac{\partial C_{\gamma\eta}}{\partial a_{mn}}]\\
&+\sum_{\alpha,\beta}\frac{\partial C_{\alpha\beta}}{\partial a_{ij}}[\sum_{s,t}F^{\alpha\beta,st}\frac{\partial a^{st}}{\partial a_{mn}}+\sum_{s,\eta}F^{\alpha\beta,s\eta}\frac{\partial B_{s\eta}}{\partial a_{mn}}+\sum_{\gamma,t}F^{\alpha\beta,\gamma t}\frac{\partial B_{t\gamma}}{\partial a_{mn}}+\sum_{\gamma,\eta}F^{\alpha\beta,\gamma\eta}\frac{\partial C_{\gamma\eta}}{\partial a_{mn}}],\end{aligned}$$

and the derivative of G in the second term in (3.14)

$$\begin{aligned}
\frac{\partial^2 G}{\partial a_{ij}\partial b_{m\eta}} &= \sum_{k,\beta} F^{k\beta}\frac{\partial^2 B_{k\beta}}{\partial a_{ij}\partial b_{m\eta}} + \sum_{\alpha,l} F^{\alpha l}\frac{\partial^2 B_{l\alpha}}{\partial a_{ij}\partial b_{m\eta}} + \sum_{\alpha,\beta} F^{\alpha\beta}\frac{\partial^2 C_{\alpha\beta}}{\partial a_{ij}\partial b_{m\eta}} \\
&+ \sum_{k,l}\frac{\partial a^{kl}}{\partial a_{ij}}[\sum_{s,\zeta} F^{kl,s\zeta}\frac{\partial B_{s\zeta}}{\partial b_{m\eta}} + \sum_{\xi,t} F^{kl,\xi t}\frac{\partial B_{t\xi}}{\partial b_{m\eta}} + \sum_{\xi,\zeta} F^{kl,\xi\zeta}\frac{\partial C_{\xi\zeta}}{\partial b_{m\eta}}] \\
&+ \sum_{k,\beta}\frac{\partial B_{k\beta}}{\partial a_{ij}}[\sum_{s,\zeta} F^{k\beta,s\zeta}\frac{\partial B_{s\zeta}}{\partial b_{m\eta}} + \sum_{\xi,t} F^{k\beta,\xi t}\frac{\partial B_{t\xi}}{\partial b_{m\eta}} + \sum_{\xi,\zeta} F^{k\beta,\xi\zeta}\frac{\partial C_{\xi\zeta}}{\partial b_{m\eta}}] \\
&+ \sum_{\alpha,l}\frac{\partial B_{l\alpha}}{\partial a_{ij}}[\sum_{s,\zeta} F^{\alpha l,s\zeta}\frac{\partial B_{s\zeta}}{\partial b_{m\eta}} + \sum_{\xi,t} F^{\alpha l,\xi t}\frac{\partial B_{t\xi}}{\partial b_{m\eta}} + \sum_{\xi,\zeta} F^{\alpha l,\xi\zeta}\frac{\partial C_{\xi\zeta}}{\partial b_{m\eta}}] \\
&+ \sum_{\alpha,\beta}\frac{\partial C_{\alpha\beta}}{\partial a_{ij}}[\sum_{s,\zeta} F^{\alpha\beta,s\zeta}\frac{\partial B_{s\zeta}}{\partial b_{m\eta}} + \sum_{\xi,t} F^{\alpha\beta,\xi t}\frac{\partial B_{t\xi}}{\partial b_{m\eta}} + \sum_{\xi,\zeta} F^{\alpha\beta,\xi\zeta}\frac{\partial C_{\xi\zeta}}{\partial b_{m\eta}}].
\end{aligned}$$

From (3.16), we can get the derivative of G in the seventh term of (3.14)

$$\begin{aligned}
\frac{\partial^2 G}{\partial b_{k\beta}\partial b_{l\alpha}} &= \sum_{\gamma,\eta} F^{\gamma\eta}\frac{\partial^2 C_{\gamma\eta}}{\partial b_{k\beta}\partial b_{l\alpha}} \\
&+ \sum_{m,\eta}\frac{\partial B_{m\eta}}{\partial b_{k\beta}}[\sum_{s,\zeta} F^{m\eta,s\zeta}\frac{\partial B_{s\zeta}}{\partial b_{l\alpha}} + \sum_{\xi,t} F^{m\eta,\xi t}\frac{\partial B_{t\xi}}{\partial b_{l\alpha}} + \sum_{\xi,\zeta} F^{m\eta,\xi\zeta}\frac{\partial C_{\xi\zeta}}{\partial b_{l\alpha}}] \\
&+ \sum_{\gamma,n}\frac{\partial B_{n\gamma}}{\partial b_{k\beta}}[\sum_{s,\zeta} F^{\gamma n,s\zeta}\frac{\partial B_{s\zeta}}{\partial b_{l\alpha}} + \sum_{\xi,t} F^{\gamma n,\xi t}\frac{\partial B_{t\xi}}{\partial b_{l\alpha}} + \sum_{\xi,\zeta} F^{\gamma n,\xi\zeta}\frac{\partial C_{\xi\zeta}}{\partial b_{l\alpha}}] \\
&+ \sum_{\gamma,\eta}\frac{\partial C_{\gamma\eta}}{\partial b_{k\beta}}[\sum_{s,\zeta} F^{\gamma\eta,s\zeta}\frac{\partial B_{s\zeta}}{\partial b_{l\alpha}} + \sum_{\xi,t} F^{\gamma\eta,\xi t}\frac{\partial B_{t\xi}}{\partial b_{l\alpha}} + \sum_{\xi,\zeta} F^{\gamma\eta,\xi\zeta}\frac{\partial C_{\xi\zeta}}{\partial b_{l\alpha}}].
\end{aligned}$$

So we denote

$$\widetilde{X}_{kl} = \sum_{i,j}\frac{\partial a^{kl}}{\partial a_{ij}}X_{ij}, \widetilde{X}_{\beta k} = \widetilde{X}_{k\beta} = \sum_{i,j}\frac{\partial B_{k\beta}}{\partial a_{ij}}X_{ij}, \widetilde{X}_{\alpha\beta} = \sum_{i,j}\frac{\partial C_{\alpha\beta}}{\partial a_{ij}}X_{ij}, \tag{3.17}$$

$$\widetilde{Y}_{kl} = 0, \widetilde{Y}_{\beta k} = \widetilde{Y}_{k\beta} = \sum_{m,\eta}\frac{\partial B_{k\beta}}{\partial b_{m\eta}}Y_{m\eta}, \widetilde{Y}_{\alpha\beta} = \sum_{m,\eta}\frac{\partial C_{\alpha\beta}}{\partial b_{m\eta}}Y_{m\eta}, \tag{3.18}$$

$$\widetilde{Z}_{kl} = 0, \widetilde{Z}_{\beta k} = \widetilde{Z}_{k\beta} = 0, \widetilde{Z}_{\alpha\beta} = Z_{\alpha\beta}. \tag{3.19}$$

From the above calculation, and (3.17)-(3.19), we can get the first term of (3.14)

$$
\begin{aligned}
&\sum_{i,j,m,n}\frac{\partial^2 G}{\partial a_{ij}\partial a_{mn}}X_{ij}X_{mn}\\
&\quad=\sum_{k,l}F^{kl}\sum_{i,j,m,n}\frac{\partial^2 a^{kl}}{\partial a_{ij}\partial a_{mn}}X_{ij}X_{mn}+\sum_{k,\beta}F^{k\beta}\sum_{i,j,m,n}\frac{\partial^2 B_{k\beta}}{\partial a_{ij}\partial a_{mn}}X_{ij}X_{mn}\\
&\qquad+\sum_{\alpha,l}F^{\alpha l}\sum_{i,j,m,n}\frac{\partial^2 B_{l\alpha}}{\partial a_{ij}\partial a_{mn}}X_{ij}X_{mn}+\sum_{\alpha,\beta}F^{\alpha\beta}\sum_{i,j,m,n}\frac{\partial^2 C_{\alpha\beta}}{\partial a_{ij}\partial a_{mn}}X_{ij}X_{mn}\\
&\qquad+\sum_{a,b,c,d=1}^{N}F^{ab,cd}\widetilde{X}_{ab}\widetilde{X}_{cd}\\
&\quad=2\sum_{a,b}F^{ab}\sum_{ij}a_{ij}\widetilde{X}_{ia}\widetilde{X}_{jb}+\sum_{a,b,c,d=1}^{N}F^{ab,cd}\widetilde{X}_{ab}\widetilde{X}_{cd},
\end{aligned}\tag{3.20}
$$

the second term of (3.14)

$$
\begin{aligned}
&\sum_{i,j,m,\eta}\frac{\partial^2 G}{\partial a_{ij}\partial b_{m\eta}}X_{ij}Y_{m\eta}\\
&\quad=\sum_{k,\beta}F^{k\beta}\sum_{i,j,m,\eta}\frac{\partial^2 B_{k\beta}}{\partial a_{ij}\partial b_{m\eta}}X_{ij}Y_{m\eta}+\sum_{\alpha,l}F^{\alpha l}\sum_{i,j,m,\eta}\frac{\partial^2 B_{l\alpha}}{\partial a_{ij}\partial b_{m\eta}}X_{ij}Y_{m\eta}\\
&\qquad+\sum_{\alpha,\beta}F^{\alpha\beta}\sum_{i,j,m,\eta}\frac{\partial^2 C_{\alpha\beta}}{\partial a_{ij}\partial b_{m\eta}}X_{ij}Y_{m\eta}+\sum_{a,b,c,d=1}^{N}F^{ab,cd}\widetilde{X}_{ab}\widetilde{Y}_{cd}\\
&\quad=\sum_{k,\beta}F^{k\beta}\sum_{i,j}a_{ij}\widetilde{X}_{ik}\widetilde{Y}_{j\beta}+\sum_{\alpha,l}F^{\alpha l}\sum_{i,j}a_{ij}\widetilde{X}_{il}\widetilde{Y}_{j\alpha}\\
&\qquad+\sum_{\alpha,\beta}F^{\alpha\beta}\sum_{i,j}a_{ij}(\widetilde{X}_{i\alpha}\widetilde{Y}_{j\beta}+\widetilde{X}_{i\beta}\widetilde{Y}_{j\alpha})+\sum_{a,b,c,d=1}^{N}F^{ab,cd}\widetilde{X}_{ab}\widetilde{Y}_{cd}\\
&\quad=\sum_{a,b}F^{ab}\sum_{i,j}a_{ij}(\widetilde{X}_{ia}\widetilde{Y}_{jb}+\widetilde{X}_{ib}\widetilde{Y}_{ja})+\sum_{a,b,c,d=1}^{N}F^{ab,cd}\widetilde{X}_{ab}\widetilde{Y}_{cd},
\end{aligned}\tag{3.21}
$$

and the seventh term of (3.14)

$$
\begin{aligned}
\sum_{k,\beta,l,\alpha}\frac{\partial^2 G}{\partial b_{k\beta}\partial b_{l\alpha}}Y_{k\beta}Y_{l\alpha}&=\sum_{\gamma,\eta}F^{\gamma\eta}\sum_{k,\beta,l,\alpha}\frac{\partial^2 C_{\gamma\eta}}{\partial b_{k\beta}\partial b_{l\alpha}}Y_{k\beta}Y_{l\alpha}+\sum_{a,b,c,d=1}^{N}F^{ab,cd}\widetilde{Y}_{ab}\widetilde{Y}_{cd}\\
&=2\sum_{\gamma,\eta}F^{\gamma\eta}\sum_{i,j}a_{ij}\widetilde{Y}_{i\gamma}\widetilde{Y}_{j\eta}+\sum_{a,b,c,d=1}^{N}F^{ab,cd}\widetilde{Y}_{ab}\widetilde{Y}_{cd}\\
&=2\sum_{a,b}F^{ab}\sum_{i,j}a_{ij}\widetilde{Y}_{ia}\widetilde{Y}_{jb}+\sum_{a,b,c,d=1}^{N}F^{ab,cd}\widetilde{Y}_{ab}\widetilde{Y}_{cd}.
\end{aligned}\tag{3.22}
$$

Also we obtain the third-sixth terms in (3.14)

$$\sum_{i,j,\gamma,\eta} \frac{\partial^2 G}{\partial a_{ij}\partial c_{\gamma\eta}} X_{ij} Z_{\gamma\eta} = \sum_{a,b,c,d=1}^{N} F^{ab,cd} \widetilde{X}_{ab} \widetilde{Z}_{cd}, \tag{3.23}$$

$$\sum_{i,j,\gamma} \frac{\partial^2 G}{\partial a_{ij}\partial p_{\gamma}} X_{ij} X_{\gamma} = \sum_{a,b=1}^{N} \sum_{\gamma} F^{ab,p_\gamma} \widetilde{X}_{ab} X_{\gamma}, \tag{3.24}$$

$$\sum_{i,j} \frac{\partial^2 G}{\partial a_{ij}\partial u} X_{ij} Y = \sum_{a,b=1}^{N} F^{ab,u} \widetilde{X}_{ab} Y, \tag{3.25}$$

$$\sum_{i,j,k} \frac{\partial^2 G}{\partial a_{ij}\partial x_{k}} X_{ij} Z_{k} = \sum_{a,b=1}^{N} \sum_{k} F^{ab,x_k} \widetilde{X}_{ab} Z_{k}, \tag{3.26}$$

and the eighth-eleventh terms in (3.14)

$$\sum_{k,\beta,\xi,\zeta} \frac{\partial^2 G}{\partial b_{k\beta}\partial c_{\xi\zeta}} Y_{k\beta} Z_{\xi\zeta} = \sum_{a,b,c,d=1}^{N} F^{ab,cd} \widetilde{Y}_{ab} \widetilde{Z}_{cd}, \tag{3.27}$$

$$\sum_{k,\beta,\xi} \frac{\partial^2 G}{\partial b_{k\beta}\partial p_{\zeta}} Y_{k\beta} X_{\zeta} = \sum_{a,b=1}^{N} \sum_{\zeta} F^{ab,p_\zeta} \widetilde{Y}_{ab} X_{\zeta}, \tag{3.28}$$

$$\sum_{k,\beta} \frac{\partial^2 G}{\partial b_{k\beta}\partial u} Y_{k\beta} Y = \sum_{a,b=1}^{N} F^{ab,u} \widetilde{Y}_{ab} Y, \tag{3.29}$$

$$\sum_{k,\beta,i} \frac{\partial^2 G}{\partial b_{k\beta}\partial x_{i}} Y_{k\beta} Z_{i} = \sum_{a,b=1}^{N} \sum_{i} F^{ab,x_i} \widetilde{Y}_{ab} Z_{i}. \tag{3.30}$$

So let $\widetilde{X} = ((\widetilde{X}_{ab} + \widetilde{Y}_{ab} + \widetilde{Z}_{ab}), (X_\alpha), Y, (Z_i))$, then we can obtain (3.13). Also the equivalence holds.

4. Structure condition and the proof of theorem 1.2

In this section, we prove Theorem 1.2 using a strong maximum principle and Lemma 3.3. Also Corollary 1.4 holds directly from the proof.

We denote $\mathcal{S}^n$ to be the set of all real symmetric $n \times n$ matrices, and denote $\mathcal{S}^n_+ \in \mathcal{S}^n$ to be the set of all positive definite symmetric $n \times n$ matrices. Let $\mathbb{O}_n$ be the space consisting all $n \times n$ orthogonal matrices and $I_{N''}$ be the $N'' \times N''$ identity matrix. We define

$$\mathcal{S}_{N'-1} = \{ \begin{pmatrix} Q \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} Q^T & Qb \\ b^T Q^T & c \end{pmatrix} \in \mathcal{S}^N \Big| \forall b \in \mathbb{R}^{N' \times N''}, \forall c \in \mathcal{S}^{N''}, \forall Q \in \mathbb{O}_{N'}, \forall B \in \mathcal{S}^{N'-1} \},$$

and for given $Q \in \mathbb{O}_{N'}$

$$\mathcal{S}_{N'-1}(Q)=\{\begin{pmatrix} Q\begin{pmatrix}0 & 0\\ 0 & B\end{pmatrix}Q^T & Qb\\ b^TQ^T & c\end{pmatrix}\in\mathcal{S}^N\Big|\forall b\in\mathbb{R}^{N'\times N''},\forall c\in\mathcal{S}^{N''},\forall B\in\mathcal{S}^{N'-1}\}$$

$$=\{\begin{pmatrix}Q & 0\\ 0 & I_{N''}\end{pmatrix}\begin{pmatrix}\begin{pmatrix}0 & 0\\ 0 & B\end{pmatrix} & b\\ b^T & c\end{pmatrix}\begin{pmatrix}Q^T & 0\\ 0 & I_{N''}\end{pmatrix}\Big|\forall b\in\mathbb{R}^{N'\times N''},\forall c\in\mathcal{S}^{N''},\forall B\in\mathcal{S}^{N'-1}\}.$$

Therefore $\mathcal{S}_{N'-1}(Q)\subset\mathcal{S}_{N'-1}\subset\mathcal{S}^N$. For any (p',x'') fixed and $Q\in\mathbb{O}_{N'}$, $(A,p'',u,x')\in\mathcal{S}_{N'-1}(Q)\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}$, we set

$$X_F^*=((F^{ab}(A,p,u,x)),F^{p_{N'+1}},\cdots,F^{p_N},F^u,F^{x_1},\cdots,F^{x_{N'}})$$

as a vector in $\mathcal{S}^N\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}$. Set

$$\Gamma^{\perp}_{X_F^*}=\{\widetilde{X}\in\mathcal{S}_{N'-1}(Q)\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}|\left\langle\widetilde{X},X_F^*\right\rangle=0\}.\tag{4.1}$$

Let $B\in\mathcal{S}^{N'-1}$, $A=B^{-1}$, and

$$\widetilde{B}=\begin{pmatrix}0 & 0\\ 0 & B\end{pmatrix},\ \widetilde{A}=\begin{pmatrix}0 & 0\\ 0 & A\end{pmatrix}.$$

For any given $Q\in\mathbb{O}_{N'}$ and $\widetilde{X}=((X_{ab}),(X_\alpha),Y,(Z_i))\in\mathcal{S}_{N'-1}(Q)\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}$, we define a quadratic form

$$\begin{aligned}Q^*(\widetilde{X},\widetilde{X})=&\sum_{a,b,c,d=1}^{N}F^{ab,cd}X_{ab}X_{cd}+2\sum_{a,b=1}^{N}\sum_{k,l=1}^{N'}F^{ab}[Q\widetilde{A}Q^T]_{kl}X_{ka}X_{lb}\\&+2\sum_{a,b=1}^{N}\sum_{\alpha=N'+1}^{N}F^{ab,p_\alpha}X_{ab}X_\alpha+2\sum_{a,b=1}^{N}F^{ab,u}X_{ab}Y+2\sum_{a,b=1}^{N}\sum_{i=1}^{N'}F^{ab,x_i}X_{ab}Z_i\\&+\sum_{\alpha,\beta=N'+1}^{N}F^{p_\alpha,p_\beta}X_\alpha X_\beta+2\sum_{\alpha=N'+1}^{N}F^{p_\alpha,u}X_\alpha Y+2\sum_{\alpha=N'+1}^{N}\sum_{i=1}^{N'}F^{p_\alpha,x_i}X_\alpha Z_i\\&+F^{u,u}Y^2+2\sum_{i=1}^{N'}F^{u,x_i}YZ_i+\sum_{i,j=1}^{N'}F^{x_i,x_j}Z_iZ_j,\end{aligned}\tag{4.2}$$

where the derivative functions of F are evaluated at $(\begin{pmatrix}Q\widetilde{B}Q^T & Qb\\ b^TQ^T & c\end{pmatrix},p,u,x)$.

From lemma 3.3, we can get

Lemma 4.1. *If F satisfies condition (1.4), then for each (p',x'')*

$$F(\begin{pmatrix}0 & b\\ b^T & c\end{pmatrix},p,u,x)\ \textit{is locally convex in}\ (c,p'',u,x'),\ \textit{and}\ Q^*(\widetilde{X},\widetilde{X})\geqslant 0,\forall\widetilde{X}\in\Gamma^{\perp}_{X_F^*},\tag{4.3}$$

where Q^* is defined in (4.2).

Proof. Taking $\varepsilon>0$ small enough such that $a=Q\begin{pmatrix}\varepsilon & 0\\ 0 & B+\varepsilon I_{N'-1}\end{pmatrix}Q^T$ is invertible, and using (3.13), where $\widetilde{X}=((X_{ab}),(X_\alpha),Y,(Z_i))\in\Gamma^{\perp}_{X_F^*}$, then we can obtain (4.3) when $\varepsilon\to 0$.

Theorem 1.2 is a direct consequence of the following theorem and Lemma 4.1.

Theorem 4.2. *Suppose Ω is a domain in $\mathbb{R}^N = \mathbb{R}^{N'} \times \mathbb{R}^{N''}$ and $F(A,p,u,x) \in C^{2,1}(\mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega)$ satisfies (1.2) and (1.4). Let $u \in C^{3,1}(\Omega)$ is a partial convex solution of (1.1). If $W(x) = (u_{ij}(x))_{N' \times N'}$ attains minimum rank l at certain point $z_0 \in \Omega$, then there is a neighborhood $\mathcal{O}$ of z_0 and a positive constant C independent of ϕ (defined in (3.2)), such that*

$$\sum_{a,b=1}^{N} F^{ab}\phi_{ab} \leqslant C(\phi + |\nabla\phi|), \quad \forall x \in \mathcal{O}. \tag{4.4}$$

In turn, $W(x)$is of constant rank in $\mathcal{O}$.

Proof of Theorem 4.2. Let $u \in C^{3,1}(\Omega)$ be a partial convex solution of equation (1.1) and $W(x) = (u_{ij}(x))_{N' \times N'}$. For each $z_0 \in \Omega$ where W attains minimal rank l. We may assume $l \leqslant N' - 1$, otherwise there is nothing to prove. As in the previous section, we pick an open neighborhood $\mathcal{O}$ of z_0, and for any $x \in \mathcal{O}$, let $G = \{N' - l + 1, \cdots, N'\}$ and $B = \{1, 2, \cdots, N' - l\}$ which means good terms and bad ones in indices for eigenvalues of $W(x)$ respectively.

Setting ϕ as (3.2), then we see from Proposition 3.1 that

$$\phi \in C^{1,1}(\mathcal{O}), \quad \phi(x) \geqslant 0, \quad \phi(z_0) = 0,$$

and there is a constant $C > 0$ such that for all $x \in \mathcal{O}$,

$$\frac{1}{C}\sigma_1(B)(x) \leqslant \phi(x) \leqslant C\sigma_1(B)(x), \quad \frac{1}{C}\sigma_1(B)(x) \leqslant \sigma_{l+1}(W(x)) \leqslant C\sigma_1(B)(x). \tag{4.5}$$

We shall fix a point $x \in \mathcal{O}$ and prove (4.4) at x. For each $x \in \mathcal{O}$ fixed, we rotate coordinate $e_1, \cdots, e_{N'}$ such that the matrix $u_{ij}, i, j = 1, \cdots, N'$ is diagonal and without loss of generality we assume $u_{11} \leqslant u_{22} \leqslant \cdots \leqslant u_{N'N'}$. Then there is a positive constant $C > 0$ depending only on $\|u\|_{C^{3,1}}$ and $\mathcal{O}$, such that $u_{N'N'} \geqslant \cdots \geqslant u_{N'-l+1N'-l+1} \geqslant C > 0$ for all $x \in \mathcal{O}$. Without confusion we will also simply denote $B = \{u_{11}, \cdots, u_{N'-lN'-l}\}$ and $G = \{u_{N'-l+1N'-l+1}, \cdots, u_{N'N'}\}$. Note that for any $\delta > 0$, we may choose $\mathcal{O}$ small enough such that $u_{jj} < \delta$ for all $j \in B$ and $x \in \mathcal{O}$.

Again, as in section 3, we will avoid to deal with $\sigma_{l+1}(W) = 0$. By considering $W_\varepsilon = W + \varepsilon I$, and $u_\varepsilon(x) = u(x) + \frac{\varepsilon}{2}|x'|^2$ for $\varepsilon > 0$ sufficient small. Thus $u_\varepsilon(x)$ satisfies equation

$$F(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x) = R_\varepsilon(x), \tag{4.6}$$

where $R_\varepsilon(x) = F(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x) - F(D^2u, Du, u, x)$. Since $u \in C^{3,1}$; we have,

$$|R_\varepsilon(x)| \leqslant C\varepsilon, \quad |\nabla R_\varepsilon(x)| \leqslant C\varepsilon, \quad \left|\nabla^2 R_\varepsilon(x)\right| \leqslant C\varepsilon, \quad \forall x \in \mathcal{O}. \tag{4.7}$$

We will work on equation (4.6) to obtain differential inequality (4.4) for $\phi_\varepsilon(x)$ defined in (3.3) with constant C_1, C_2 independent of ε. Theorem 4.2 would follow by letting $\varepsilon \to 0$. In the following, we may as well omit the subindex ε for convenience.

We note that by (3.4), we have

$$\varepsilon \leqslant C\phi(x), \quad \forall x \in \mathcal{O},$$

with $R(x)$ under control as follows,

$$\left|D^j R_\varepsilon(x)\right| \leqslant C\varepsilon, \text{ for all } j = 0, 1, 2, \text{ and for all } x \in \mathcal{O}. \tag{4.8}$$

Differentiate (4.6) one time in x_i for $i \in B$, then we can get

$$\sum_{a,b=1}^{N} F^{ab} u_{abi} + \sum_{a=1}^{N} F^{p_a} u_{ai} + F^u u_i + F^{x_i} = O(\phi),$$

i.e.

$$\sum_{a,b=N'-l+1}^{N} F^{ab} u_{abi} + \sum_{a=N'+1}^{N} F^{p_a} u_{ai} + F^u u_i + F^{x_i} = O(\sum_{i,j\in B} |\nabla u_{ij}| + \phi). \tag{4.9}$$

Differentiate (4.6) twice in x_i for $i \in B$, then we obtain

$$\begin{aligned}
&\sum_{a,b=1}^{N} F^{ab} u_{abii} + \sum_{a,b=1}^{N} u_{abi}[\sum_{c,d=1}^{N} F^{ab,cd} u_{cdi} + \sum_{c=1}^{N} F^{ab,p_c} u_{ci} + F^{ab,u} u_i + F^{ab,x_i}] \\
&+ \sum_{a=1}^{N} F^{p_a} u_{aii} + \sum_{a=1}^{N} u_{ai}[\sum_{c,d=1}^{N} F^{p_a,cd} u_{cdi} + \sum_{c=1}^{N} F^{p_a,p_c} u_{ci} + F^{p_a,u} u_i + F^{p_a,x_i}] \\
&+ F^u u_{ii} + u_i[\sum_{c,d=1}^{N} F^{u,cd} u_{cdi} + \sum_{c=1}^{N} F^{u,p_c} u_{ci} + F^{u,u} u_i + F^{u,x_i}] \\
&+ \sum_{c,d=1}^{N} F^{x_i,cd} u_{cdi} + \sum_{c=1}^{N} F^{x_i,p_c} u_{ci} + F^{x_i,u} u_i + F^{x_i,x_i} = O(\phi),
\end{aligned} \tag{4.10}$$

i.e.

$$\begin{aligned}
&\sum_{a,b=1}^{N} F^{ab} u_{abii} + \sum_{a,b,c,d=N'-l+1}^{N} F^{ab,cd} u_{abi} u_{cdi} + 2 \sum_{a,b=N'-l+1}^{N} \sum_{c=N'+1}^{N} F^{ab,p_c} u_{abi} u_{ci} \\
&+ 2 \sum_{a,b=N'-l+1}^{N} F^{ab,u} u_{abi} u_i + 2 \sum_{a,b=N'-l+1}^{N} F^{ab,x_i} u_{abi} + \sum_{a,c=N'+1}^{N} F^{p_a,p_c} u_{ai} u_{ci} \\
&+ 2 \sum_{a=N'+1}^{N} F^{p_a,u} u_{ai} u_i + 2 \sum_{a=N'+1}^{N} F^{p_a,x_i} u_{ai} + F^{u,u} u_i^2 + 2F^{u,x_i} u_i + F^{x_i,x_i} \\
=&O(\sum_{i,j\in B} |\nabla u_{ij}| + \phi).
\end{aligned} \tag{4.11}$$

So for each $i \in B$, let

$$\begin{aligned}
J_i =& \sum_{a,b,c,d=N'-l+1}^{N} F^{ab,cd} u_{abi} u_{cdi} + 2 \sum_{a,b=N'-l+1}^{N} \sum_{c=N'+1}^{N} F^{ab,p_c} u_{abi} u_{ci} + 2 \sum_{a,b=N'-l+1}^{N} F^{ab,u} u_{abi} u_i \\
&+ 2 \sum_{j\in G} \frac{1}{u_{jj}} \sum_{a,b=N'-l+1}^{N} F^{ab} u_{ija} u_{ijb} + 2 \sum_{a,b=N'-l+1}^{N} F^{ab,x_i} u_{abi} + \sum_{a,c=N'+1}^{N} F^{p_a,p_c} u_{ai} u_{ci} \\
&+ 2 \sum_{a=N'+1}^{N} F^{p_a,u} u_{ai} u_i + 2 \sum_{a=N'+1}^{N} F^{p_a,x_i} u_{ai} + F^{u,u} u_i^2 + 2F^{u,x_i} u_i + F^{x_i,x_i}.
\end{aligned} \tag{4.12}$$

Substitute (4.11) and (4.12) into (3.5), then we obtain

$$
\begin{aligned}
\sum_{ab=1}^{N} F^{ab}\phi_{ab} \quad =& -\sum_{i\in B}[\sigma_l(G)+\frac{\sigma_1^2(B\,|i)-\sigma_2(B\,|i)}{\sigma_1^2(B)}]J_i\\
&-\frac{1}{\sigma_1^3(B)}\sum_{i\in B}\sum_{ab=1}^{N}F^{ab}[\sigma_1(B)u_{iia}-u_{ii}\sum_{j\in B}u_{jja}][\sigma_1(B)u_{iib}-u_{ii}\sum_{j\in B}u_{jjb}]\\
&-\frac{1}{\sigma_1(B)}\sum_{\substack{i,j\in B\\ i\neq j}}\sum_{ab=1}^{N}F^{ab}u_{ija}u_{ijb}\\
&+O(\sum_{i,j\in B}|\nabla u_{ij}|+\phi).
\end{aligned}
\tag{4.13}
$$

By condition (1.4), since $u\in C^{3,1}$, so $F^{ab}\in C^{0,1}$. For $\overline{\mathcal{O}}\subset\Omega$, there exists a constant $\delta_0>0$, such that

$$
(F^{ab})\geqslant\delta_0 I_N,\quad \forall x\in\mathcal{O}.
\tag{4.14}
$$

Case(i): $l=0$. Then $G=\emptyset$ and

$$
\begin{aligned}
J_i=&\sum_{a,b,c,d=N'+1}^{N}F^{ab,cd}(D^2u,Du,u,x)u_{abi}u_{cdi}+2\sum_{a,b=N'+1}^{N}\sum_{c=N'+1}^{N}F^{ab,p_c}u_{abi}u_{ci}\\
&+2\sum_{a,b=N'+1}^{N}F^{ab,u}u_{abi}u_i+2\sum_{a,b=N'+1}^{N}F^{ab,x_i}u_{abi}+\sum_{a,c=N'+1}^{N}F^{p_a,p_c}u_{ai}u_{ci}\\
&+2\sum_{a=N'+1}^{N}F^{p_a,u}u_{ai}u_i+2\sum_{a=N'+1}^{N}F^{p_a,x_i}u_{ai}+F^{u,u}u_i^2+2F^{u,x_i}u_i+F^{x_i,x_i},
\end{aligned}
\tag{4.15}
$$

where all the derivative functions of F are evaluated at (D^2u,Du,u,x). Since $F\in C^{2,1}$ and $\|W(x)\|_{C^0}=O(\phi)$, by Taylor formula and condition (4.3), we can get

$$
\begin{aligned}
J_i=&O(\phi)+\sum_{a,b,c,d=N'+1}^{N}F^{ab,cd}u_{abi}u_{cdi}+2\sum_{a,b=N'+1}^{N}\sum_{c=N'+1}^{N}F^{ab,p_c}u_{abi}u_{ci}\\
&+2\sum_{a,b=N'+1}^{N}F^{ab,u}u_{abi}u_i+2\sum_{a,b=N'+1}^{N}F^{ab,x_i}u_{abi}+\sum_{a,c=N'+1}^{N}F^{p_a,p_c}u_{ai}u_{ci}\\
&+2\sum_{a=N'+1}^{N}F^{p_a,u}u_{ai}u_i+2\sum_{a=N'+1}^{N}F^{p_a,x_i}u_{ai}+F^{u,u}u_i^2+2F^{u,x_i}u_i+F^{x_i,x_i}\\
\geqslant&-C\phi,
\end{aligned}
\tag{4.16}
$$

where all the derivative functions of F are evaluated at $(\begin{pmatrix}0 & (u_{k\alpha})\\ (u_{\alpha k}) & (u_{\alpha\beta})\end{pmatrix},p,u,x)$.

Case(ii): $1 \leqslant l \leqslant N'-1$
Now we set $X_{ab}=0$ for $a \in B$ or $b \in B$,

$$X_{N'N'} = u_{N'N'i} - \frac{1}{F^{N'N'}}[\sum_{a,b=N'-l+1}^{N} F^{ab}u_{abi} + \sum_{a=N'+1}^{N} F^{p_a}u_{ai} + F^{u}u_i + F^{x_i}], \tag{4.17}$$

$X_{ab} = u_{abi}$ otherwise, $Y = u_i$ and $Z_k = \delta_{ki}$. We can verify that $(X_{ab}) \in S_{N'-1}(I_{N'})$ and $\widetilde{X} = ((X_{ab}),(X_\alpha),Y,(Z_i)) \in \Gamma^{\perp}_{X_F^*}$. Again by condition (4.3), we infer that

$$J_i \geqslant -C(\sum_{i,j\in B}|\nabla u_{ij}| + \phi), \tag{4.18}$$

since $C > \sigma_l(G) + \frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)} > 0$, thus we obtain

$$\begin{aligned}\sum_{a,b=1}^{N} F^{ab}\phi_{ab} &\leqslant C(\sum_{i,j\in B}|\nabla u_{ij}|+\phi)\\ &\quad -\frac{1}{\sigma_1^3(B)}\sum_{i\in B}\sum_{a,b=1}^{N}F^{ab}[\sigma_1(B)u_{iia}-u_{ii}\sum_{j\in B}u_{jja}][\sigma_1(B)u_{iib}-u_{ii}\sum_{j\in B}u_{jjb}]\\ &\quad -\frac{1}{\sigma_1(B)}\sum_{\substack{i,j\in B\\ i\neq j}}\sum_{a,b=1}^{N}F^{ab}u_{ija}u_{ijb}\\ &\leqslant C(\sum_{i,j\in B}|\nabla u_{ij}|+\phi) - \frac{\delta_0}{\sigma_1^3(B)}\sum_{i\in B}\sum_{a=1}^{N}\widetilde{V}_{ia}^2 - \frac{\delta_0}{\sigma_1(B)}\sum_{\substack{i,j\in B\\ i\neq j}}\sum_{a=1}^{N}u_{ija}^2,\end{aligned} \tag{4.19}$$

where $\widetilde{V}_{ia} = \sigma_1(B)u_{iia} - u_{ii}\sum_{j\in B}u_{jja}$. Referring to Lemma 3.3 in [2], we can control the term $\sum_{i,j\in B}|\nabla u_{ij}|$ by the rest terms on the right hand side in (4.19) and $\phi + |\nabla\phi|$ where

$$\phi_a = O(\phi) + \sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B\,|i\,)-\sigma_2(B\,|i\,)}{\sigma_1^2(B)}]u_{iia}. \tag{4.20}$$

So there exist positive constants C_1,C_2 independent of ε, such that

$$\sum_{ab=1}^{N} F^{ab}\phi_{ab} \leqslant C_1(\phi+|\nabla\phi|) - C_2\sum_{i,j\in B}|\nabla u_{ij}|, \quad \forall x \in \mathcal{O}. \tag{4.21}$$

Taking $\varepsilon \to 0$, (4.19) is proved for u. By the Strong Maximum Principle, $\phi(x) \equiv 0$ in $\mathcal{O}$; and W is of constant rank in $\mathcal{O}$. The proof of Theorem 4.2 is completed.

Remark 4.3. In the above proof, we have used a weak condition (4.3). Also we can directly use the condition (1.4), i.e.(3.13). We set $X_{ab}=0$ for $a \in B$ or $b \in B$, $X_{ab} = u_{abi}$ otherwise, $Y = u_i$ and $Z_k = \delta_{ki}$. Then we have $\widetilde{X} = ((X_{ab}),(X_\alpha),Y,(Z_i)) \in \mathcal{S}^N \times \mathbb{R}^{N''} \times \mathbb{R} \times \mathbb{R}^{N'}$, and by (3.13), $J_i \geqslant 0$ for every $i \in B$. So (4.19) holds. As above, Theorem 4.2 holds.

Remark 4.4. In particular, for $N'=1$, we only need the following structure condition

$$F(\begin{pmatrix} 0 & b \\ b^T & c \end{pmatrix}, p', p'', u, x', x'') \text{ is locally convex in } (c, p'', u, x'), \tag{4.22}$$

then we have $(u_{ij})_{N'\times N'}$ is of constant rank in Ω. Since when $N'=1$, so the minimum rank l has only two cases: $l=1$ and $l=0$. If $l=1$ we are done; and if $l=0$, (4.16) and (4.19) holds by condition (4.22). Then the result holds as the proof of Theorem 4.2.

5. THE PROOF OF THEOREM 1.5

In this section we give the proof of Theorem 1.5. It is similar to the proof of Theorem 1.2 only some minor modifications.

Following the notations of Theorem 1.5, suppose $W(x,t_0)=(u_{ij}(x,t_0))_{N'\times N'}$ attains minimal rank $l=l(t_0)$ at some point $z_0\in\Omega$. We may assume $l\leqslant N'-1$, otherwise there is nothing to prove. As in the section 4, there is a neighborhood $\mathcal{O}\times(t_0-\delta,t_0+\delta]$ of (z_0,t_0) instead of $\mathcal{O}$, such that $u_{N'N'}\geqslant\cdots\geqslant u_{N'-l+1N'-l+1}\geqslant C>0$ for all $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta]$, and we can denote $B=\{u_{11},\cdots,u_{N'-lN'-l}\}$ and $G=\{u_{N'-l+1N'-l+1},\cdots,u_{N'N'}\}$. If $t_0=T$, the neighborhood should be $\mathcal{O}\times(t_0-\delta,t_0]$.

Setting ϕ as (3.2) (where $W(x,t)$ instead of $W(x)$), then we see from Proposition 3.1 that

$$\phi\in C^{1,1}(\mathcal{O}\times(t_0-\delta,t_0+\delta]),\quad \phi(x,t)\geqslant 0,\quad \phi(z_0,t_0)=0,$$

Also when we choose $\mathcal{O}$ and $\delta>0$ small enough, the corresponding (3.4), (4.5) and (4.8) hold. Then Theorem 1.5 is a consequence of the following theorem and the method of continuity.

Theorem 5.1. *Suppose Ω is a domain in $\mathbb{R}^N=\mathbb{R}^{N'}\times\mathbb{R}^{N''}$ and $F(A,p,u,x,t)\in C^{2,1}(\mathcal{S}^N\times\mathbb{R}^N\times\mathbb{R}\times\Omega\times(0,T])$ satisfies (1.2) for each t and (1.8). Let $u\in C^{3,1}$ is a partial convex solution of (1.9). For each $t_0\in(0,T]$, if $W(x,t_0)=(u_{ij}(x,t_0))_{N'\times N'}$ attains minimum rank l at some point $z_0\in\Omega$, then there is a neighborhood $\mathcal{O}\times(t_0-\delta,t_0+\delta]$ of (z_0,t_0) as above and a positive constant C independent of ϕ (defined in (3.2)), such that*

$$\sum_{ab=1}^{N}F^{ab}\phi_{ab}(x,t)-\phi_t(x,t)\leqslant C(\phi(x,t)+|\nabla\phi(x,t)|),\quad \forall(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta]. \tag{5.1}$$

In turn, $W(x,t)$ has constant rank l in $\mathcal{O}\times(t_0-\delta,t_0]$, where $l=l(t_0)$.

Proof of Theorem 5.1.The proof is similar to the proof of Theorem 4.2, so we only give the main process of the proof.

With $u_t=F(D^2u,Du,u,x,t)$, using the same notations as above and the proof of Theorem 4.2, we have $u_\varepsilon(x,t)=u(x,t)+\frac{\varepsilon}{2}|x'|^2$ for $\varepsilon>0$ sufficient small. Thus $u_\varepsilon(x,t)$ satisfies equation

$$(u_\varepsilon)_t=F(D^2u_\varepsilon,Du_\varepsilon,u_\varepsilon,x,t)-R_\varepsilon(x,t), \tag{5.2}$$

where $R_\varepsilon(x,t)=F(D^2u_\varepsilon,Du_\varepsilon,u_\varepsilon,x,t)-F(D^2u,Du,u,x,t)$.

As in the proof of Theorem 4.2, we omit the subindex ε.

Differentiate (5.2) one time in x_i for $i \in B$, then we can get

$$\sum_{a,b=1}^{N} F^{ab} u_{abi} + \sum_{a=1}^{N} F^{p_a} u_{ai} + F^u u_i + F^{x_i} = O(\phi) + u_{i,t} \ ,$$

i.e.

$$\sum_{a,b=N'-l+1}^{N} F^{ab} u_{abi} + \sum_{a=N'+1}^{N} F^{p_a} u_{ai} + F^u u_i + F^{x_i} = O(\sum_{i,j\in B} |\nabla u_{ij}| + \phi) + u_{i,t} \ . \tag{5.3}$$

Differentiate (5.2) twice in x_i for $i \in B$, then we can get

$$\begin{aligned}
&\sum_{a,b=1}^{N} F^{ab} u_{abii} + \sum_{a,b=1}^{N} u_{abi}[\sum_{c,d=1}^{N} F^{ab,cd} u_{cdi} + \sum_{c=1}^{N} F^{ab,p_c} u_{ci} + F^{ab,u} u_i + F^{ab,x_i}] \\
&+ \sum_{a=1}^{N} F^{p_a} u_{aii} + \sum_{a=1}^{N} u_{ai}[\sum_{c,d=1}^{N} F^{p_a,cd} u_{cdi} + \sum_{c=1}^{N} F^{p_a,p_c} u_{ci} + F^{p_a,u} u_i + F^{p_a,x_i}] \\
&+ F^u u_{ii} + u_i[\sum_{c,d=1}^{N} F^{u,cd} u_{cdi} + \sum_{c=1}^{N} F^{u,p_c} u_{ci} + F^{u,u} u_i + F^{u,x_i}] \\
&+ \sum_{c,d=1}^{N} F^{x_i,cd} u_{cdi} + \sum_{c=1}^{N} F^{x_i,p_c} u_{ci} + F^{x_i,u} u_i + F^{x_i,x_i} = O(\phi) + u_{ii,t} \ ,
\end{aligned} \tag{5.4}$$

i.e.

$$\begin{aligned}
&\sum_{a,b=1}^{N} F^{ab} u_{abii} + \sum_{a,b,c,d=N'-l+1}^{N} F^{ab,cd} u_{abi} u_{cdi} + 2 \sum_{a,b=N'-l+1}^{N} \sum_{c=N'+1}^{N} F^{ab,p_c} u_{abi} u_{ci} \\
&+ 2 \sum_{a,b=N'-l+1}^{N} F^{ab,u} u_{abi} u_i + 2 \sum_{a,b=N'-l+1}^{N} F^{ab,x_i} u_{abi} + \sum_{a,c=N'+1}^{N} F^{p_a,p_c} u_{ai} u_{ci} \\
&+ 2 \sum_{a=N'+1}^{N} F^{p_a,u} u_{ai} u_i + 2 \sum_{a=N'+1}^{N} F^{p_a,x_i} u_{ai} + F^{u,u} u_i^2 + 2F^{u,x_i} u_i + F^{x_i,x_i} \\
=&O(\sum_{i,j\in B} |\nabla u_{ij}| + \phi) + u_{ii,t} \ .
\end{aligned} \tag{5.5}$$

We denote that

$$\phi_t = \sum_{i,j=1}^{N'} \frac{\partial \phi}{\partial u_{ij}} u_{ij,t} = \sum_{i=1}^{N'} \frac{\partial \phi}{\partial u_{ii}} u_{ii,t} \ ,$$

so we can obtain from (3.5), (4.12) and (5.5),

$$
\begin{aligned}
\sum_{ab=1}^{N} F^{ab}\phi_{ab}(x,t)-\phi_t(x,t) = &-\sum_{i\in B}[\sigma_l(G)+\frac{\sigma_1^2(B\,|\,i)-\sigma_2(B\,|\,i)}{\sigma_1^2(B)}]J_i\\
&-\frac{1}{\sigma_1^3(B)}\sum_{i\in B}\sum_{ab=1}^{N}F^{ab}[\sigma_1(B)u_{iia}-u_{ii}\sum_{j\in B}u_{jja}][\sigma_1(B)u_{iib}-u_{ii}\sum_{j\in B}u_{jjb}]\\
&-\frac{1}{\sigma_1(B)}\sum_{\substack{i,j\in B\\ i\neq j}}\sum_{ab=1}^{N}F^{ab}u_{ija}u_{ijb}\\
&+O(\sum_{i,j\in B}|\nabla u_{ij}|+\phi).
\end{aligned}
\tag{5.6}
$$

Now the right hand side of (5.6) is the same as the right hand side of (4.13). From Remark 4.3, we set $X_{ab}=0$ for $a\in B$ or $b\in B$, $X_{ab}=u_{abi}$ otherwise, $Y=u_i$ and $Z_k=\delta_{ki}$. Then we have $\widetilde{X}=((X_{ab}),(X_\alpha),Y,(Z_i))\in\mathcal{S}^N\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}$, and by (3.13), $J_i\geqslant 0$ for every $i\in B$. So (4.19) holds. A similar analysis as in the proof of Theorem 4.2 for the right hand side of equation (4.19) yields

$$
\sum_{ab=1}^{N} F^{ab}\phi_{ab}(x,t)-\phi_t(x,t)\leqslant C_1(\phi(x,t)+|\nabla\phi(x,t)|)-C_2\sum_{i,j\in B}|\nabla u_{ij}|,
\tag{5.7}
$$

where the positive constants C_1,C_2 independent of ε, and $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta]$. Then $W(x,t)$ has a constant rank l for each $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0]$ by the Strong Maximum Principle for parabolic equations. Theorem 5.1 holds.

6. DISCUSSION OF STRUCTURE CONDITION

In this section, we discuss the condition (4.3) and (1.4).

For any given $Q\in\mathbb{O}_{N'}$, we define

$$
\widetilde{F_Q}(A,b,c,p'',u,x')=F(\begin{pmatrix} Q\begin{pmatrix}0&0\\0&A^{-1}\end{pmatrix}Q^T & Q\begin{pmatrix}0&0\\0&A^{-1}\end{pmatrix}b\\ b^T\begin{pmatrix}0&0\\0&A^{-1}\end{pmatrix}Q^T & c+b^T\begin{pmatrix}0&0\\0&A^{-1}\end{pmatrix}b\end{pmatrix},p,u,x),
$$

for $(A,b,c,p'',u,x')\in\mathcal{S}_+^{N'-1}\times\mathbb{R}^{N'\times N''}\times\mathcal{S}^{N''}\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}$ and fixed $(p',x'')\in\mathbb{R}^{N'}\times\mathbb{R}^{N''}$. Condition (1.4) implies the following condition

$$
\widetilde{F}_Q(A,b,c,p'',u,x') \text{ is locally convex in } (A,b,c,p'',u,x'),
\tag{6.1}
$$

for any fixed $N'\times N'$ orthogonal matrix Q.

Proposition 6.1. *Let $Q\in\mathbb{O}_{N'}$. The condition (6.1)is equivalent to*

$$
Q^*(\widetilde{X},\widetilde{X})\geqslant 0,
\tag{6.2}
$$

for any $\widetilde{X}=((X_{ab}),(X_\alpha),Y,(Z_i))\in\mathcal{S}_{N'-1}(Q)\times\mathbb{R}^{N''}\times\mathbb{R}\times\mathbb{R}^{N'}$, where Q^* is defined in (4.2).

Proof. By approximating, Proposition 6.1 holds from Lemma 3.2.

Remark 6.2. Condition (1.4) is equivalent to (3.13), and (1.4) implies (6.1) for any fixed $N' \times N'$ orthogonal matrix Q. Condition (6.1) is equivalent to (6.2), and Lemma 4.1 is a consequence of Proposition 6.1. And condition (6.1) is weaker than condition (1.4).

There is a class of functions which satisfy (1.4). Through a direct calculation and using (3.13), we can get

Proposition 6.3. *If g is a non-decreasing and convex function and $F_1, \cdots, F_m$ satisfy condition (1.4), then $F = g(F_1, \cdots, F_m)$ also satisfies condition (1.4). In particular, if F_1 and F_2 are in the class, so are $F_1 + F_2$ and F_1^α(where $F_1 > 0$) for any $\alpha \geqslant 1$.*

Remark 6.4. This paper was finished in April 2009, and B. Bian and P. Guan give a better structural condition (an equivalent condition of (4.3)) in their paper "A Structural Condition for Microscopic Convexity Principle", which appears in Discrete and Continuous Dynamical Systems, Volume 28, Number 2, 2010, pp. 789-807.

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Department of Mathematics, University of Science and Technology of China, Hefei, 230026, Anhui Province, CHINA
E-mail address: cqchen@mail.ustc.edu.cn